Naomi Feldheim

Research Statement

1. Overview

My research is devoted to the interplay between Analysis and Probability. I use tools from real, complex and harmonic analysis to solve problems which arise in probability or mathematical physics. A significant part of my studies is concerned with *Gaussian stationary processes* and in partuclar *stationary Gaussian analytic functions*. These are discussed in Section 2. I am also interested in *concentration* and *discrepancy*, which are discussed in Section 3.

2. Gaussian Stationary Processes

Let $D \in \{\mathbb{R}, \mathbb{C}\}$ and $R \in \{\mathbb{R}, \mathbb{C}\}$. A random function $f : D \to R$ is **Gaussian** if all its finite marginals have centered Gaussian distribution, and it is **stationary** if its distribution is invariant with respect to all *real* shifts. Such a process, which we call a *Gaussian stationary process* (GSP), is uniquely determined by its *covariance kernel*

$$r(t) = \mathbb{E}[f(0)\overline{f(t)}] = \mathbb{E}[f(s)\overline{f(s+t)}].$$

Some of the motivation for studying GSPs stems from their utility in modeling "stationary noise" such as brain transmission, radiation, ocean waves and internet traffic. They also appear as the scaling limit of various models in statistical mechanics and as components in more complex models in physics. In addition, the zeroes of Gaussian stationary processes form a family of natural point processes, which have been extensively studied in mathematics, physics and engineering (see e.g. [1, 14]). Special attention was drawn lately to Gaussian **analytic** functions, whose zeroes form a rich family of rigid repulsive point processes, used to model electric charge (see [26]).

While, in principle, every property of a GSP could be described in terms of its covariance kernel, the task of providing this description could be daunting and the description itself could be rather complex. A prominent theme in my research is that many global properties of GSPs are better understood through *the spectral measure* ρ, which is the inverse Fourier transform of the covariance:

$$r(t) = \widehat{\rho}(t) = \int_{\mathbb{R}} e^{-i\lambda t}\, d\rho(\lambda).$$

The spectral point of view, combined with tools from probability, harmonic and complex analysis, yields fairly precise and elegant answers to long-lasting questions about zeroes of GSPs, such as:

(1) **Density of zeroes** (goes back to Wiener, Kac): What is the mean number of zeroes in a region, and does a "law of large numbers" hold?
(2) **Fluctuations and limit theorems** (goes back to Cramér, Cuzick): What determines the order of magnitude of the variance? do zeroes obey a "central limit theorem"?
(3) **Persistence** (goes back to Slepian): What is the asymptotic behavior of the probability that a GSP has no zeroes in a large region?
(4) **Concentration** (goes back to Rice): What is the probability that the number of zeroes in a large region deviates from its mean significantly?

Next, I describe my contribution to each of these aspects, closing the section with an outline of near-future plans.

2.1. **Density of zeroes.** Wiener considered the model of a stationary Gaussian analytic function (henceforth, stationary GAF), defined on a strip $D_\Delta = \{|\mathrm{Im}\, z| < \Delta\}$, with $0 < \Delta \leq \infty$. This model includes some famous examples, such as

$$(1) \qquad \sum_{n=0}^{\infty} w_n(\zeta_{2n}\cos(\lambda_n z) + \zeta_{2n+1}\sin(\lambda_n z)), \qquad \sum_{n\in\mathbb{Z}} \zeta_n \frac{\sin(\pi(z-n))}{z-n}, \qquad e^{-z^2/2}\sum_{n=0}^{\infty}\zeta_n \frac{z^n}{\sqrt{n!}},$$

where $\zeta_n \sim \mathcal{N}_{\mathbb{C}}(0,1)$ are i.i.d., and in the left-most one $w_n, \lambda_n \in \mathbb{R}$ are given (with $\sum_n w_n^2 e^{\lambda_n y} < \infty$ for all $|y| < \Delta$, to ensure convergence in D_Δ).

In [37, Ch. X] Wiener proved a law of large numbers for the zeroes of stationary GAFs in a horizontal strip, whose main assumption is the existence of square-summable spectral density, providing also an explicit formula for the limit in this case. In [20] I show that Wiener's law of large numbers and formula for the expected horizontal density of zeroes hold for *any* stationary GAF, and I further provide a criterion for the limit to be deterministic (equal to its expectation). Denoting by $n_f(A)$ the number of zeroes of f in a sub-domain $A \subset D_\Delta$, the main result in [20] is:

Theorem 1. *(a) A.s. for all intervals $I \subset (-\Delta, \Delta)$ the limit $\nu_f(I) := \lim_{T\to\infty} \frac{n_f([0,T]\times I)}{T}$ exists.*
(b) ν_f is non-random $\iff$ the spectral measure is continuous or consists of exactly one atom.
(c) If ν_f is non-random, then it has the continuous density $\frac{1}{4\pi}\frac{d}{dy}\left\{\frac{\psi'(y)}{\psi(y)}\right\}$ where $\psi(y) = \mathbb{E}\left[|f(iy)|^2\right]$.

The limit measure ν_f is a (potentially random) locally-finite measure, which describes *the horizontal density of zeroes*. The main tools to prove its existence (item (a)) and non-randomness (item (b)) come from ergodic theory and measure theory, while the explicit computation in (c) uses a version of Edelman-Kostlan's formula.

Symmetry around the real axis: collapsing zeroes phenomenon. A natural counterpart of GAFs are *symmetric GAFs*, that is, Gaussian analytic functions $f : D \to \mathbb{C}$ on a domain $D \subset \mathbb{C}$ which posses a symmetry around the real axis: a.s., $\forall z \in D: \ \overline{f(z)} = f(\overline{z})$ (notice that "Gaussian" here means that a marginal of n values has Gaussian distribution in $\mathbb{R}^{2n}$). These random functions are represented by random series with *real* i.i.d. Gaussian coefficients; examples are given by taking real coefficients $\zeta_n \sim \mathcal{N}_{\mathbb{R}}(0,1)$ in (1). One may ask:

> How many zeroes of such a random function are real? How do other zeroes behave, compared to the case of complex coefficients?

For the special case of *random polynomials with real coefficients*, Kac [27] computed the "density of zeroes" on the real line, while Shepp-Vanderbei [40] computed the density in the rest of the complex plane. In [20] we generalize these works by giving a formula for the expected number of zeroes of any symmetric GAF in any domain.

Theorem 2. *If f is a symmetric GAF on D and $K(z,w) = \mathbb{E}[f(z)\overline{f(w)}]$, then for any sub-domain $A \subseteq D$ the expected number of zeroes in A is given by:*

$$\mathbb{E}n_f(A) = \frac{1}{4\pi}\int_A \Delta \log\left(K(z,z) + \sqrt{K(z,z)^2 - |K(z,\overline{z})|^2}\right) dz,$$

where Δ is the distributional Laplacian operator.

In contrast, for a classical (non-symmetric) GAF one can compute the density of zeroes using Edelman-Kostlan's formula $\mathbb{E}n_f(A) = \frac{1}{4\pi}\int_A \Delta \log K(z,z)\ dz$ [26, Ch. 2.4]. A comparison between the two yields an interesting relation between a GAF $f_{\mathbb{C}}$ and a symmetric GAF $f_{\mathbb{R}}$ which share the same covariance kernel: away from the real line, they have very similar densities of zeroes; while as one approaches the real line, more and more zeroes of $f_{\mathbb{R}}$ "collapse" to the real line (i.e. the density of complex zeroes tends to zero, and real zeroes compensate for them). Special cases of this behavior stirred interest among physicists who considered them as models for condensation, see Schehr-Majumadar [39]. The phenomenon has recently attracted the attention of Vanderbei [46], who gave an alternative proof for Theorem 2 and used it to analyze specific examples.

Using Theorem 2 we prove in [20] an analogue of Theorem 1 for symmetric stationary GAFs.

2.2. **Fluctuations and central limit theorems.** Estimating the variance of the number of zeroes in a large region is generally much harder than computing its expectation. In the case of real GSPs this was done using involved computations of Wiener-Itô integrals [15, 42]. Sophisticated Kac-Rice computations were carried out for zeroes of algebraic polynomials [32] and trigonometric polynomials [19], and diagram methods were used for the Fock-Bargmann planar GAF [35, 43].

All these methods rely on the success of estimating the covariance of the process, and of relating it to the covariance between zeroes. The works described below introduce a new spectral, rather elementary, approach in order to obtain an explicit formula for the variance in two models: zeroes of a stationary GAFs [21] and winding of SGPs in the plane [11].

Complex zeroes. Continuing in the setup of Section 2.1, let f be a stationary GAF in the strip D_Δ with spectral measure ρ. We assume that ρ does not consist of exactly one atom, as this case is degenerate. Fix $-\Delta < a < b < \Delta$ and denote by $V(T)$ the variance of the number of zeroes of f in $[0,T]\times[a,b]$. In [21] we provide the following estimates for $V(T)$.

Theorem 3. *(i) There exists $C, c > 0$ such that $cT \le V(T) \le CT^2$ for all large enough T.*
(ii) $V(T) \asymp T$ if the following L^2-condition holds:

$$\forall y \in \{2a, 2b\}: \quad \int_{\mathbb{R}} |r(x+iy)|^2 dx < \infty, \quad \int_{\mathbb{R}} |r''(x+iy)|^2 dx < \infty. \tag{2}$$

(iii) $V(T) \gg T$ if condition (2) fails, and a certain regularity assumption holds.
(iv) $V(T) \asymp T^2$ if and only if the spectral measure of f contains an atom.

These results give insight to the global behavior of zeroes of GAFs:

- *Zeroes are never "super-concentrated"*, since $V(T)$ is always at least linear in T.
- Condition (2) is nearly optimal to describe *linear variance*. This is precisely the condition given in [42] for the case of real SGPs.
- *Rigid zeroes:* When $V(T) \gg T$, the process exhibits a strong dependence between distant zeroes. This is extremal when the spectral measure has an atom.

Winding number. Typically a function $f:\mathbb{R}\to\mathbb{C}$ has no zeros whatsoever. The appropriate analogue to the number of zeroes in this case is the *winding number* i.e., *the increment of the* *argument*. For a continuously differentiable function the winding may be expressed by $\Delta(T) = \int_0^T \frac{f'}{f}(t)dt$. One motivation for studying winding of GSPs is its usage as a model for entanglement of polymers and the movement of a charged particle under a random magnetic field [18]. In a joint work with Jeremiah Buckley [11], we investigate this quantity for a GSP f, establishing and extending predictions by Le-Doussal, Etzioni and Horovitz [18]. We prove an analogue of Theorem 3, only that now $V(T) = \mathrm{var}\,[\Delta(T)]$ represents the variance of the winding. We deduce that the winding number shares the same qualitative properties mentioned earlier for complex zeroes. Moreover, we give an explicit formula for $V(T)$, which reduces to $V(T) = T\int_{\mathbb{R}} \frac{r'^2}{1-r^2} + O(1)$ when $r(t) = r(-t)$ and $r' \in L^2(\mathbb{R})$. At last, we prove that in case $r, r'' \in L^2$ (which matches condition (2)), a CLT holds: $\frac{\Delta(T)-\mathbb{E}[\Delta(T)]}{\sqrt{\mathrm{var}\,[\Delta(T)]}} \to \mathcal{N}_{\mathbb{R}}(0,1)$ in law.

Relation with the spectral measure. Although the spectral measure is hardly apparent in the formulation of these results, it plays a major role in the proofs. For complex zeroes, we develop an expression for $V(T)$ in terms of the spectral measure, before applying tools from harmonic and asymptotic analysis to infer its asymptotic growth. To obtain the CLT for the winding we modify a spectral method which appeared in the study of real zeroes, see [15].

2.3. **Persistence probability.** The *persistence probability* of a random function $f : \mathbb{R} \to \mathbb{R}$ is defined by $P_f(N) := \mathbb{P}(f > 0 \text{ on } [0, N])$. This quantity is of importance in applications in noise theory, statistical physics and other models within probability (see recent surveys in mathematics [5] and physics [13], and workshops [49, 50]). In a seminal 1962 paper [41] Slepian asked:

> What are possible asymptotic behaviors of the persistence probability?
> What features of the covariance kernel determine this behavior?

These basic questions remained, at large, unanswered for many years, as the best known bounds were relatively crude [36]. To some extent this may be due to the fact that they are ill posed: our recent research indicates that the persistence is mainly controlled by the behavior of the spectral measure near the origin. This paradigm was initiated by our work with O. Feldheim [22], in which we prove that the existence of spectral density in a neighborhood of the origin which is bounded away from zero and infinity, is morally enough to ensure that the persistence is roughly exponential (i.e. $\exists c_1, c_2 > 0$ s.t. $0 < c_1 \le \frac{-\log P_f(N)}{N} \le c_2$). This was generalized and expanded in a recent joint work with O. Feldheim and S. Nitzan [24], in which we classify the decay of $P_f(N)$ in terms of the spectral measure's behavior near the origin. We prove:

Theorem 4. *Suppose that the spectral measure has density $w(\lambda)$ such that $\int \lambda^\delta w(\lambda)d\lambda < \infty$ for some $\delta > 0$, and $c_1\lambda^\alpha \le w(\lambda) \le c_2\lambda^\alpha$ for all λ in a neighborhood of 0 (and some $\alpha > -1$, $c_1, c_2 > 0$). Then:*

$$\log P_f(N) \begin{cases} \asymp -N^{1+\alpha}\log N, & \alpha < 0 \\ \asymp -N, & \alpha = 0 \\ \lesssim -N\log N, & \alpha > 0. \end{cases}$$

Moreover, if $w(\lambda)$ vanishes on an interval containing 0, then $P_f(N) \le e^{-CN^2}$. If in addition the tail has polynomial decay (or heavier), then $P_f(N) \le e^{-e^{CN}}$.

This is the first general bound on long range persistence which applies to sign-changing and non-summable covariance kernels. The case of $\alpha = 0$, which corresponds to *exponential decay*, generalizes a bound obtained by Antezana-Buckley-Marzo-Olsen [4] for the specific case of the sinc-kernel $r(t) = \frac{\sin(\pi t)}{\pi t}$. The case $\alpha < 0$ (exploding spectrum), which corresponds to slower than exponential decay, has been previously known only for the special case of non-negative covariance kernels (Dembo-Mukherjee [16]). The case $\alpha > 0$ (vanishing spectrum), which yields fast decay, matches lower bounds obtained for discrete time by Krishna-Krishnapur [28]. The criterion for persistence smaller than e^{-cN^2} establishes the *spectral gap conjecture* which has been circulating for a few years (see [28]). However, the existence of persistence as tiny as $e^{-e^{CN}}$ was surprising even for the experts, and shows that generally no matching lower bound can be given in the case $\alpha > 0$. Indeed, in this case one must take into account the interplay between spectral behavior at 0 and infinity.

2.4. **Concentration.** Once expectation and fluctuations have been established, it is desirable to obtain finer estimate for the number of zeroes, in the form of *large deviations estimates*; That is, to estimate the probability that the number of zeroes in a long interval $[0, T]$ differs from its mean by more than ηT (for a given $\eta > 0$). In particular, does *exponential concentration* hold, i.e., does this probability decay exponentially in T (with a constant depending on η)?

This question is not covered by the current theorems of large deviations for Gaussian stationary processes (established in [17] and [10]), since the event in question is too delicate for their topology. Nonetheless, exponential concentration was shown for related models which have additional regularity, such as complex zeroes of the Fock-Bargmann GAF [44] and nodal lines of spherical harmonics [34]. However, until recently, concentration for real zeroes was not known even for a single non-trivial example (see [45]). Attempts at this problem yielded estimates of high moments [15] and of the tail [31] of the number of zeroes in a given interval. However, to extend these results towards concentration would require lower bounds on the determinant of nearly singular covariance matrices, at a level of accuracy which seems out of reach.

Recently, together with R. Basu, A. Dembo and O. Zeitouni [6], we bypassed this difficulty by moving to the count of zeroes in the complex plane (assuming an analytic extension exists). Thereby zero counts are replaced with more regular integrals of $\log |f(z)|$. In order to estimate these, we provide sharp bounds on fractional moments of products of many dependent Gaussian variables (a non-trivial task even for integer moments). In this way we establish concentration of zeroes for a class of smooth GSPs with summable correlations:

Theorem 5. *Let f be a GSP with compactly supported spectral measure and integrable covariance (ie. $\int |r(t)|dt < \infty$). Then, for some $C < \infty$ and $c(\cdot) > 0$,*

$$\mathbb{P}\Big(|N_f([0,T]) - \alpha T| \geq \eta T \Big) \leq C e^{-c(\eta)T}, \qquad \forall \eta > 0, \ \ T < \infty . \tag{3}$$

More generally, (3) *holds if $\int e^{2\Delta|\lambda|} d\rho(\lambda) < \infty$ and $\int_{\mathbb{R}} |r(t + i\Delta_0)|dt < \infty$ for some $0 \leq \Delta_0 < \Delta$.*

2.5. **Future work.** Recent progress in the theory of GSPs inspires several conjectures and promising directions for future research. Listed below are several questions which I intend to consider.

Density of zeroes. It remains to understand the horizontal density of zeroes for a GAF whose spectral measure contains atoms. In this case the limiting horizontal density of zeroes exists, but is non-deterministic (recall Theorem 1).

Fluctuations. Is it true that the variance of the number of zeroes of real GSPs is always at least linear (an analogue of Theorem 3)? Find an accessible way of estimating the variance, so that one could infer simple conditions for intermediate growth rates. Obtain a single variant of Theorem 3 which encompasses real zeroes and complex zeroes alike.

Central limit theorems. What is the limit law of the normalized number of zeroes in a large region? For real zeroes a Gaussian law (CLT) was obtained in case of linear variance [15, 42], yet this remains to be proved for complex zeroes. A more interesting challenge would be to obtain limit theorems in cases of non-linear variance, which may possibly be non-Gaussian.

Persistence in one-dimension. First, establish lower bounds in Theorem 4 for vanishing spectrum (the case $\alpha > 0$). In particular, show that for compactly supported spectral measures the same lower bounds hold as over $\mathbb{Z}$ (tiny persistence is not possible). This would require a deep understanding of the interplay between the spectrum near 0 and near infinity.

Second, extend Theorem 4 to singular spectral measures (as a mean for gainning a more abstract and simple understanding of the cause for persistence). A promising idea of F. Nazarov and B. Jaye initiated an ongoing investigation in this direction.

Third, describe the shape of a GSP conditioned on the event of persistence: does the function tend to vanish shortly after the long positive interval? does it tend to be small / oscillatory?

Persistence in higher dimensions. Among the many analogues in high-dimensions, I would like to focus on: What is the probability that a GSP $f : \mathbb{R}^d \to \mathbb{R}$ (representing multi-parameter data [1]) has no zeroes in a "nice" large domain? What is the probability that a complex GSP $f : \mathbb{R} \to \mathbb{C}$ (representing a polymer, or the movement of an electron [18]) does not wind at all around the origin for a long time?

A more challenging type of questions is about GSPs subject to boundary conditions on some domain. In physics these are called "interface models" [13, Ch. 14] and persistence represents the presence of a "hard barrier". Usually, when the dimension is high enough, correlations decay fast and the persistence probability is easier to handle; this was done in [9] for the Gaussian free field, in [29, 30] for membrane and related models and in [38] for diffusive processes. However in low dimensions these models are much more rigid, and much less understood. The rate of decay of the persistence probability is one way to capture this rigidity, yet it is unknown even for simple examples such as the membrane model in 2 and 3 dimensions.

Concentration of zeroes. Does exponential concentration of zeroes (Theorem 5) hold for non-absolutely summable covariances? In particular, does it hold for the *sinc-kernel* $r(t) = \frac{\sin(\pi t)}{\pi t}$? I strongly believe the answer is "yes", and that such results may be retrieved using spectral methods (which are not employed to their full extent in [6]).

Universality. Can any of our results be generalized to *non-Gaussian* stationary processes?

3. Concentration and Discrepancy

This section describes several works which lie in the interface between Analysis and Probability. A theme which is common to all of these is some relation with concentration or anti-concentration phenomena. In Section 3.1 we describe a result relating the classical "small ball conjecture" from probability with discrepancy theory. Section 3.2 is devoted to a characterization of the convex Poincaré inequality. In Section 3.3 we discuss an anti-concentration distribution-free result, which relates mean and minimum of non-negative independent random variables.

3.1. Small ball inequality and binary nets.

The small ball inequality is a bound on the supremum norm achieved by any combination of dyadic haar functions of fixed volume in d-dimensions. It was first conjectured by Talagrand nearly 30 years ago, and has since found potential applications in PDE and harmonic analysis (see [7] for more details).

For a dyadic interval $I = [\frac{m}{2^k}, \frac{m+1}{2^k}] \subset [0,1]$ where m, k are integers, we associate *the Haar function* h_I which takes the value -1 on the left half of I, the value $+1$ on the right, and 0 out of I. In higher dimensions, for $R = R_1 \times \cdots \times R_d \subset [0,1]^d$ where R_i are dyadic intervals, we define $h_R(x_1, \ldots, x_d) = h_{R_1}(x_1) \cdots h_{R_d}(x_d)$. The *small ball conjecture* states that for any $d \geq 2$, $n \in \mathbb{N}$ and $\{\alpha_R\} \in \mathbb{R}$, we have:

$$n^{\frac{d-2}{2}} \cdot \Big\| \sum_{R\colon |R|=2^{-n}} \alpha_R h_R \Big\|_\infty \gtrsim 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R|,$$

where the sum is over all dyadic boxes of volume 2^{-n}. The symbol $\gtrsim$ indicates a multiplication by a constant which does not depend on n or $\{\alpha_R\}$. The *signed small ball conjecture* concerns the special case when all coefficients are $\varepsilon_R \in \{-1, +1\}$, namely $\Big\| \sum_{R\colon |R|=2^{-n}} \varepsilon_R h_R \Big\|_\infty \gtrsim n^{\frac{d}{2}}$.

Dimension $d = 2$ is the only case when the conjecture is known to hold, proved separately by Talagrand and by Temlyakov. Together with D. Bilyk [8] we give a new short proof of this result, inspired by lacunary Fourier series, which yields a novel connection with binary nets. These are "well-distributed" sets which are commonly used in numeric integration. A set of 2^m dyadic points is called a *perfect binary net* if it intersects each dyadic rectangle of area 2^{-m} in precisely one point. We prove:

Theorem 6. *Let $d = 2$, and $n \in \mathbb{N}$.*

(i) For any signs $\varepsilon_R \in \{\pm 1\}$ we have $\| \sum_{|R|=2^{-n}} \varepsilon_R h_R \|_\infty = n + 1$.

(ii) The maximal value $n + 1$ is always achieved on a perfect binary net.

(iii) Every perfect binary net is the maximal set of $\sum_{|R|=2^{-n}} \varepsilon_R h_R$ for some $\varepsilon_R \in \{\pm 1\}$.

By this we establish the first formal relation between the small ball conjecture and *discrepancy theory* (the study of "well-distributed" sets), a relation which has long been known to hold on a heuristic level. As a corollary we are able to enumerate the number of perfect binary nets of size 2^m (reproving a result from [48]). Our results extend to b-adic nets for any integer $b \geq 2$. In the future, we plan to use our new approach in order to tackle the high-dimensional case.

3.2. Convex concentration of measure.

In the last few decades much effort has been devoted to study *concentration of measure*. We say that an $\mathbb{R}^n$ valued random vector X satisfies concentration with profile $\alpha(t)$ if for any set $A \subset \mathbb{R}^n$ with $\mathbb{P}(X \in A) \geq \frac{1}{2}$ we have $\mathbb{P}(X \in A + tB_2^n) \geq 1 - \alpha(t)$, where $B_2^n = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$ is the Euclidean ball. A common approach to reach concentration is via functional inequalities, such as Poincaré inequality (the requirement that for all $f \in C^1(\mathbb{R}^n, \mathbb{R})$ we have $\mathrm{var}\,[f(X)] \leq C\mathbb{E}\|\nabla f(X)\|^2$) which yields exponential $\alpha(t)$, or log-Sobolev inequality, which yields Gaussian $\alpha(t)$. However, these inequalities are very restrictive, and one would wish to reach concentration for larger families of measures. Bobkov and Götze [12] initiated the study of the *convex Poincaré inequality*, which is simply the Poincaré property restricted to convex f. This yields exponential concentration for *convex* sets A.

Together with A. Marsiglietti, P. Nayar and J. Wang [25], we give a full characterization of one-dimensional measures satisfying the convex Poincaré inequality. We show that, in fact, this property is equivalent to the convex infimum convolution inequality suggested by Maurey [33], and is satisfied precisely by probability measures μ for which $\mu([x+h,\infty)) \leq \lambda\mu([x,\infty))$ for all $x \geq 0$ and some $h > 0$, $\lambda \in [0,1)$. This result generalizes to product measures. Nonetheless, characterizing "convex concentration" for non-product measures in $\mathbb{R}^n$ is still open.

3.3. **Mean and Minimum.** Let X, Y be two independent non-compactly supported random variables taking positive values. Clearly $\min(X,Y) \leq \frac{X+Y}{2} \leq \max(X,Y)$, so that for all $m \geq 0$,

$$\mathbb{P}\Big(\min(X,Y) > m\Big) \leq \mathbb{P}\left(\frac{X+Y}{2} > m\right) \leq \mathbb{P}\Big(\max(X,Y) > m\Big).$$

There are examples for which when m tends to infinity $\mathbb{P}\left(\frac{X+Y}{2} > m\right) \asymp \mathbb{P}\Big(\max(X,Y) > m\Big)$, but is it possible to have $\mathbb{P}\Big(\min(X,Y) > m\Big) \asymp \mathbb{P}\left(\frac{X+Y}{2} > m\right)$?

In a joint work with O. Feldheim [23] we verify a conjecture posed to us by Noga Alon by proving that the answer is "NO". More precisely, we show that for *any* independent X, Y on $(0,\infty)$, it holds that

$$\liminf_{m\to\infty} \frac{\mathbb{P}(\min(X,Y) > m)}{\mathbb{P}(X+Y > 2m)} = 0.$$

Since $\mathbb{P}(\min(X,Y) > m)$ is nothing but $\mathbb{P}(X > m)\mathbb{P}(Y > m)$, this result may be viewed as an *anti-concnetration* statement on product measures, stating, roughly, that a product measure cannot be too concentrated around the diagonal. Being *distribution-free*, it relates to a series of works about *distribution-free inequalities* which began with the "123 theorem" of Alon and Yuster [3]. In [23] we also provide a multiple-variables, weighted variant of our result in the i.i.d. case, which we conjecture to hold in general. The proof is based on general observations about the relation between a non-decreasing function and its convex minorant (later applied to the log-tail function $m \mapsto -\log P(X > m)$).

Moreover, our result has implications on a model for *evolving social groups* introduced in [2]. A goal for future research would be to obtain stronger results about this model, including its a.s. convergence. This will require additional random walk techniques.

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